# Supporting K–3 students' understanding of standard algorithms while working toward fluency

Sybilla Beckmann

Department of Mathematics
University of Georgia

Kansas MTSS Symposium September 2013

# In this session . . .

. . . we will examine some parts of these domains in the Common Core State Standards for Mathematics in grades K–3:

- Counting and Cardinality
- Operations and Algebraic Thinking
- Numbers and Operations in Base Ten

We will focus on ways of fostering understanding while supporting students to move toward fluency.

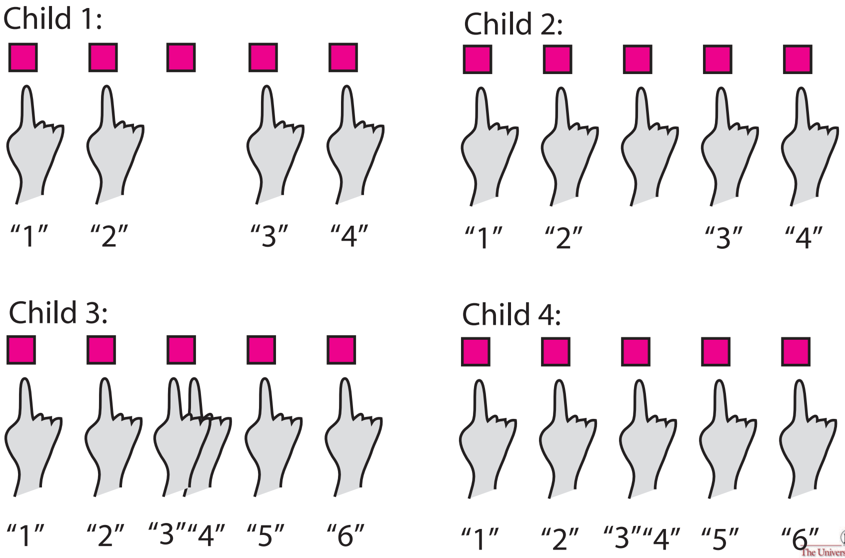
# Counting and Cardinality: Kindergarten

If a child can correctly say the first five counting numbers,

"one, two, three, four, five,"

will the child necessarily be able to determine how many blocks there are in this collection?

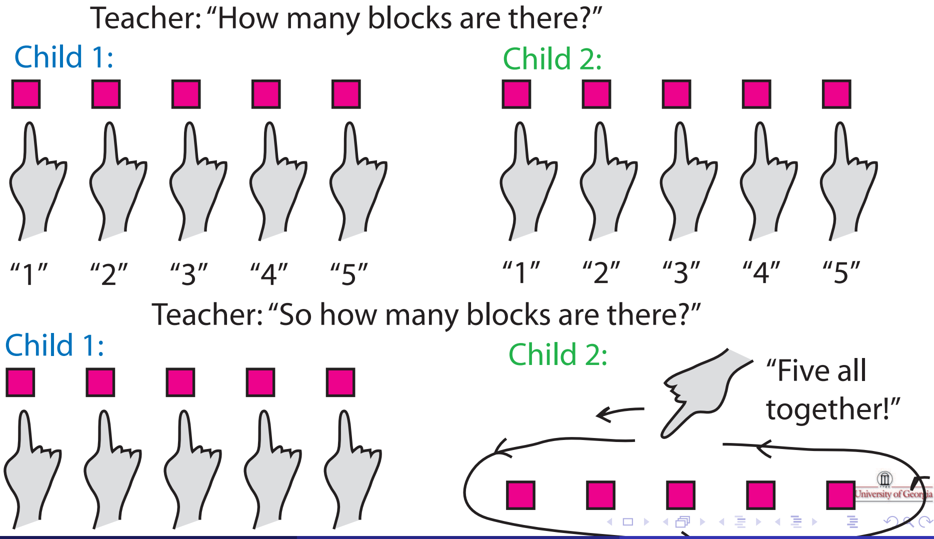
# Counting and Cardinality: Kindergarten




The University of Georgia

# Counting and Cardinality

## The last number word tells how many in all



# Operations and Algebraic Thinking, K – 5

## Summary of the domain

- Meanings of addition and subtraction (K – 2),
  multiplication and division (3 – 5)
  types of problems these operations solve
    - MP1 Make sense of problems and persevere in solving them
    - MP2 Reason abstractly and quantitatively
    - MP4 Model with mathematics

- Algebraic properties of the operations; other patterns and rules

- Single-digit additions/related subtractions;
  single digit multiplications/related divisions;

  use of properties in *learning* them, *not rotely memorizing them*
    - MP7 Look for an make use of structure
    - MP8 Look for and express regularity in repeated reasoning

The University of Georgia

# CCSS Table 1: Common types of $+-$ word problems

| | Result Unknown | Change Unknown | Start Unknown |
|---|---|---|---|
| **Add to** | Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ? | Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5 | Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5 |
| **Take from** | Five apples were on the table. I ate two apples. How many apples are on the table now? 5 - 2 = ? | Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 - ? = 3 | Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? - 2 = 3 |

| | Total Unknown | Addend Unknown | Both Addends Unknown[1] |
|---|---|---|---|
| **Put Together/ Take Apart[2]** | Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ? | Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 - 3 = ? | Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0<br>5 = 1 + 4, 5 = 4 + 1<br>5 = 2 + 3, 5 = 3 + 2 |

| | Difference Unknown | Bigger Unknown | Smaller Unknown |
|---|---|---|---|
| **Compare[3]** | ("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy?<br>("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie?<br>2 + ? = 5, 5 - 2 = ? | (Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have?<br>(Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have?<br>2 + 3 = ?, 3 + 2 = ? | (Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have?<br>(Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have?<br>5 - 3 = ?, ? + 3 = 5 |

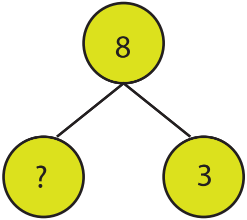
These are "change" problems that involve change over time.

Easier ones start in K, harder ones Grade 1 and up

| | Result unknown | Change unknown | Start unknown |
|---|---|---|---|
| Add to | $2+3=?$ | $2+?=3$ | $?+3=5$ |
| Take from | $5-2=?$ | $5-?=3$ | $?-2=3$ |

Kwon has some cars. He gets 3 more cars. Now he has 8 cars in all. How many cars did Kwon have before he got more?

Note: students who rely only on keywords may mistakenly *add* 3 and 8.



These problems are about amounts made up of two distinct parts.
Also known as "part-part-whole" problems.
They don't involve change over time.

Easier ones start in K, all types are Grade 1 and up

| | Total unknown | Addend unknown | Both addends unknown |
|---|---|---|---|
| Put together/ Take apart | $3 + 2 = ?$ | $3 + ? = 5$<br><br>$5 - 3 = ?$ | $5 = 0 + 5$<br>$5 = 5 + 0$<br>$5 = 1 + 4$<br>$5 = 4 + 1$<br>$5 = 2 + 3$<br>$5 = 3 + 2$ |

Grade 1 and up

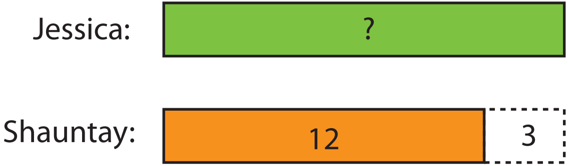
| | Difference unknown | Bigger unknown | Smaller unknown |
|---|---|---|---|
| Compare | "how many more?" wording | "more" wording | "more" wording |
| Compare | "how many fewer?" wording | "fewer/less" wording | "fewer/less" wording |

Grade 2 and up: two step problems, all types

# Compare, bigger unknown, "fewer" wording

Jessica has some cards. Shauntay has 3 fewer cards than Jessica. Shauntay has 12 cards. How many cards does Jessica have?

Note: students who rely only on keywords may mistakenly *subtract* 3 from 12.



# Links to other learning

- “Start unknown” and “change unknown” “Add to” and “take from” problems are “algebra” problems (as are some other types).
- In “take apart” situations the equal sign can’t be viewed as “calculate the answer.”

$$5 = 2 + 3$$

- Coming up: “Take apart” is necessary for level 3 addition strategies that use the associative property.

The University of Georgia

# Your turn: write and discuss $+-$ word problems

| | | | |
|---|---|---|---|
| Add to | Result unknown | Change unknown | Start unknown |
| Take from | Result unknown | Change unknown | Start unknown |
| Put together/ Take apart | Total unknown | Addend unknown | Both addends unknown |
| Compare | Difference unknown “how many more?” wording “how many fewer?” wording | Bigger unknown “more” wording “fewer/less” wording | Smaller unknown “more” wording “fewer/less” wording |

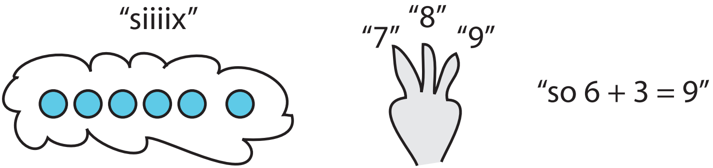
The University of Georgia

Progression of numerical strategies in working toward fluency — *not just rote memorization of the single-digit facts*:

- Level 1: count all (K)
- Level 2: count on, count on from larger, count on to subtract (Grade 1)
- Level 3: derived fact methods, especially methods that work with 10 (Grades 1, 2)

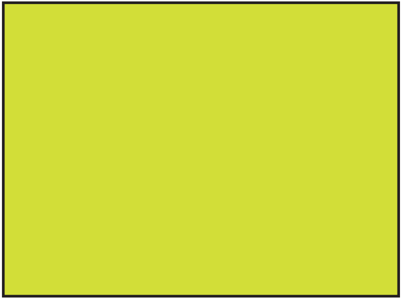
# Level 2: Counting on

What is $6 + 3$?

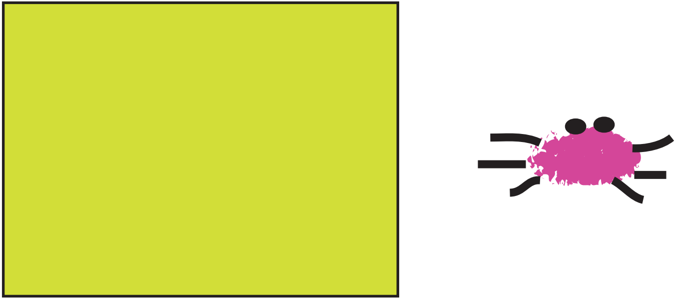


# Working towards counting on

# Working towards counting on



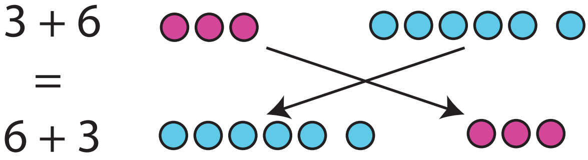
Hide them.

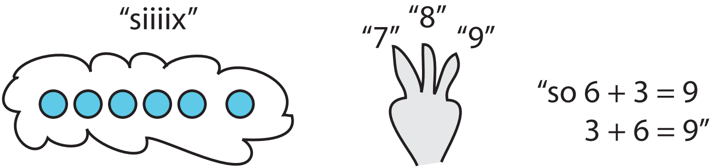


Ask: How many bugs are there altogether?

# Level 2: Applying commutativity to count on from larger

What is $3 + 6$?



# Level 2: Counting on to subtract

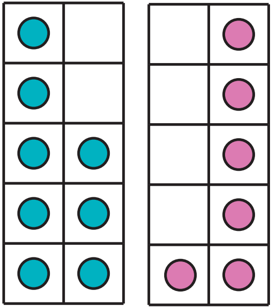
A $7 - 5 = \square$ problem: There were 7 nuts. Then a mouse ate 5. How many nuts are left? Children can also solve this by counting on from 5:


This method links subtraction and addition:

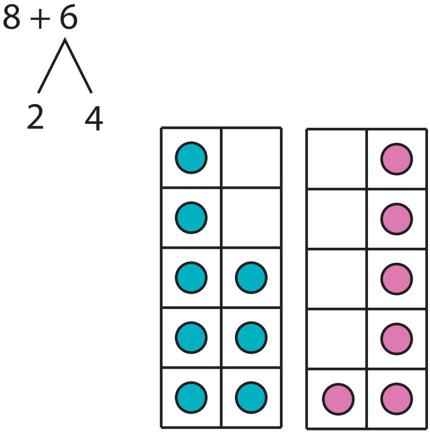
$$7 - 5 = \square \quad \leftrightarrow \quad 5 + \square = 7$$

# Level 3: Emphasizing grouping by tens

$8 + 6$

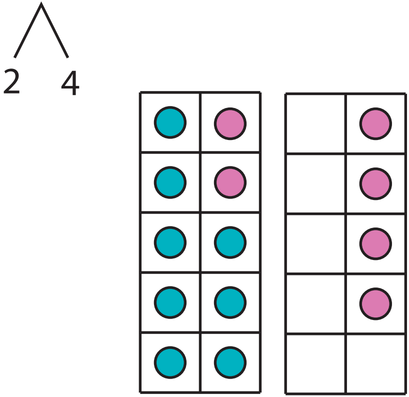


# Level 3: Emphasizing grouping by tens



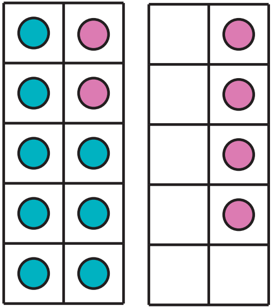
# Level 3: Emphasizing grouping by tens

$$8 + 6 = 8 + (2 + 4)$$

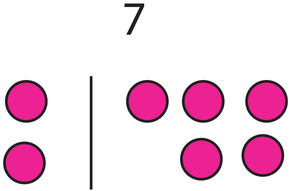


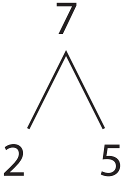
# Level 3: Emphasizing grouping by tens

$$8 + 6 = 8 + (2 + 4) = (8 + 2) + 4 = 14$$

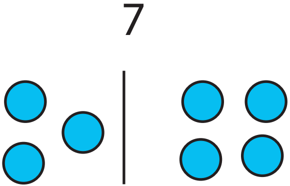


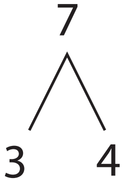
# Level 3 requires breaking numbers apart into partners




$$7 = 2 + 5$$

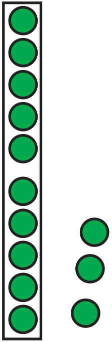


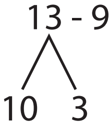

$$7 = 3 + 4$$

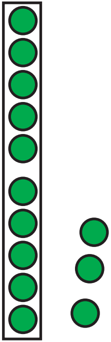
# Level 3: Emphasizing grouping by tens

13 - 9

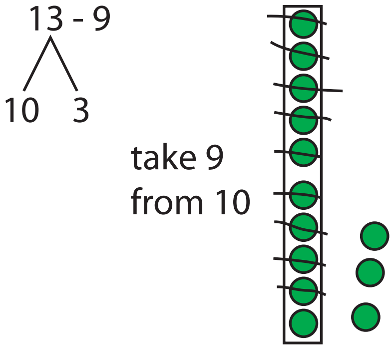


# Level 3: Emphasizing grouping by tens

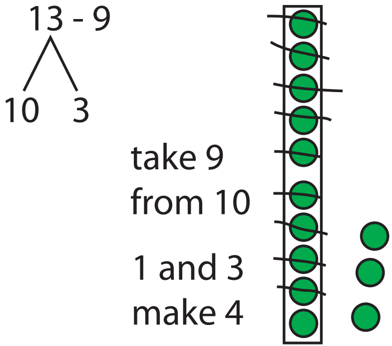


# Level 3: Emphasizing grouping by tens

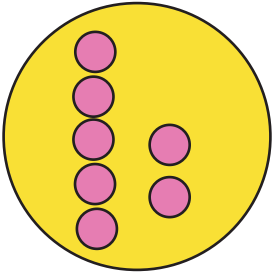


# Level 3: Emphasizing grouping by tens



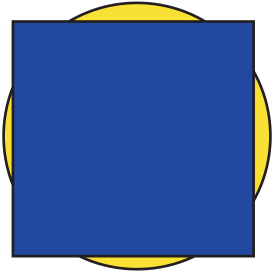
# Working towards derived methods

Show briefly:



# Working towards derived methods

Then hide:



Ask: How many are there?

# Your turn: discuss the three levels

- Level 1: count all (K)
- Level 2: count on, count on from larger, count on to subtract (Grade 1)
- Level 3: derived fact methods, especially methods that work with 10 (Grades 1, 2)

Try these methods for and $14 - 9$ and $12 - 3$:

- (Level 2) Count on to find the unknown addend
- (Level 3) Make-a-ten with the unknown addend
- (Level 3) Subtract from ten
- (Level 3) Subtract down to ten first

Discuss why these are prerequisites for Level 3:

- For each number 1 through 9, know the "partner to 10"
- For each number 11 through 19, know it as a ten and some ones
- For each number 2 through 9, know the ways to decompose it into partners

# Number and Operations in Base Ten, K – 5

## Summary of the domain

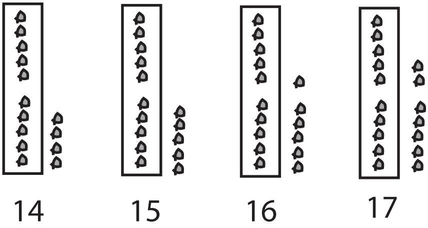
Representing, comparing, and calculating with numbers in base ten.

- Use strategies based on place value and properties of operations
- Illustrate and explain calculations with representations such as mathematical drawings
  - MP2 Reason abstractly and quantitatively
  - MP3 Construct viable arguments and critique the reasoning of others
  - MP5 Use appropriate tools strategically
  - MP6 Attend to precision
  - MP7 Look for and make use of structure
  - MP8 Look for and express regularity in repeated reasoning
- Work towards fluency with understanding

The University of Georgia

# Understanding 11 through 19 as a ten and some ones

Kindergarten: Numbers 11 through 19 are ten ones and some more ones
Grade 1: Ten ones form a unit of ten



# Some difficulties

Difficulties with spoken number words in English:

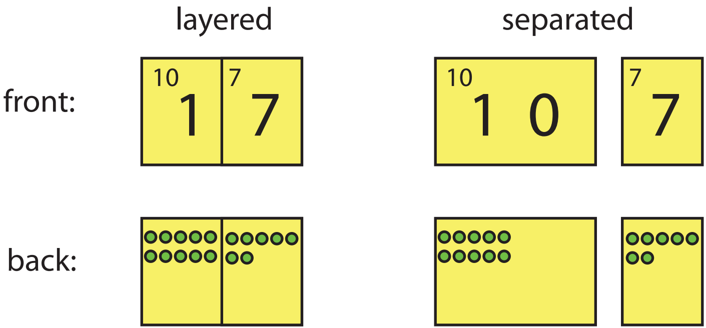
- "eleven," "twelve" do not sound like ten and one, ten and two
- "thirteen," "fourteen," etc. reverse the ones and tens
- teen words often sound like decade words: sixteen versus sixty
- "teen" may not be recognized as meaning ten

Difficulties with written numerals:

- 16 looks like "one six" and not like 1 ten and 6.

# Understanding 11 through 19 as a ten and some ones

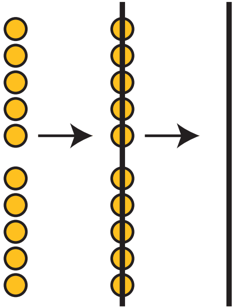
Layered place value cards



# Math drawings to show base ten structure

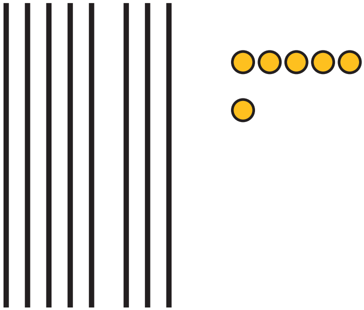
## MP7 Look for and make use of structure

drawings to support
seeing 10 ones as 1 ten



10 ones — 1 ten — 1 ten stick

86



8 tens and 6 ones

# Learning path for multi digit computation in CCSS

Please see your handout!

Phase 3: Compact methods for fluency

Phase 2: Research-based mathematically-desirable and accessible methods in the middle for understanding and growing fluency

Phase 1: Students' methods elicited for understanding but move rapidly to Phase 2

# Strategies, the standard algorithms, and written methods

Fuson, K.C. & Beckmann, S. (Fall/Winter 2012-02013). Standard algorithms in the Common Core State Standards. *National Council of of Supervisors of Mathematics Journal of Mathematics Education Leadership, 14*(2), 4 – 30. www.mathedleadership.org

Please see your handout for examples of **written methods** that implement the **standard addition algorithm** and the **standard subtraction algorithm**. We will look at those in detail, but first . . .

# Strategies, the standard algorithms, and written methods

**Computation strategies** — thoughtful approaches; the emphasis is on student sense-making.
These include special strategies for particular problems that may not generalize.

For example:

$$98 + 17 = 98 + (2 + 15) = (98 + 2) + 15 = 100 + 15 = 115$$

Or:
To multiply $8 \times 15$:
double 15 $\to$ 30, double 30 $\to$ 60, double 60 $\to$ 120

# Strategies, the standard algorithms, and written methods

**Standard algorithms** — for each operation there is a particular *mathematical approach* that is based on decomposing numbers into base-ten units and applying properties of operations to reduce a calculation to single-digit calculations together with correct place value placement.

$$
\begin{aligned}
456 + 167 &= (400 + 50 + 6) + (100 + 60 + 7) \\
&= (400 + 100) + (50 + 60) + (6 + 7) \\
&= (4 + 1)100 + (5 + 6)10 + (6 + 7)
\end{aligned}
$$

The University of Georgia

# Strategies, the standard algorithms, and written methods

To implement a standard algorithm we use a systematic **written method** for recording the steps of the algorithm.

There are variations in written methods — some are longer because they include extra steps or math drawings.

Over time the longer written methods can be abbreviated to shorter methods that allow students to achieve fluency with the standard algorithm while still being able to understand and explain the method.

# Strategies, the standard algorithms, and written methods

In the past there has been an unfortunate dichotomy suggesting that *strategy* implies understanding and *algorithm* implies no visual models, no explaining, and no understanding.

In the past, teaching *the standard algorithm* has too often meant teaching numerical steps rotely and having students memorize steps rather than understand and explain them.

The CCSS clearly do not mean for this to happen!

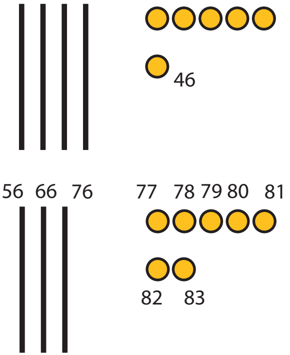
The University of Georgia

# Add by counting on with tens and ones

## A special strategy

Grade 1 numerical work side by side with mathematical drawing

$$\begin{array}{r} 46 \\ +37 \\ \hline 83 \end{array}$$

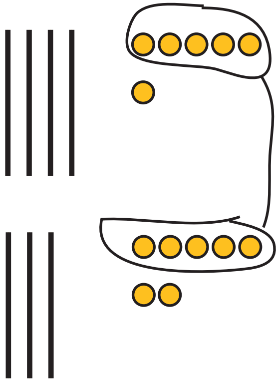


# Add tens and ones separately

## The standard algorithm

Grade 1 numerical work side by side with a mathematical drawing

```
  46
+37
 8 3
```



# Subtraction

Important: no two-digit subtraction involving both tens and ones until it is done *with regrouping*.

This is so that the error of always subtracting the smaller digit from the larger digit does not take hold.

# Understanding and explaining the standard subtraction algorithm

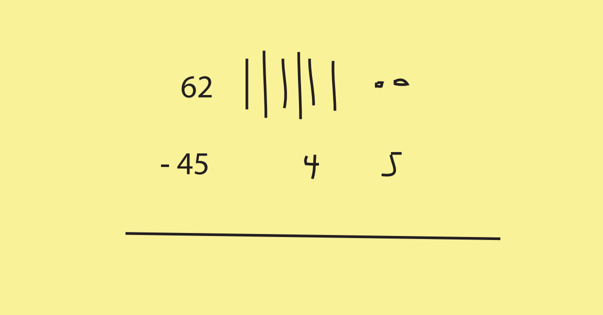
Grade 2

$$\begin{array}{r} 62 \\ -\ 45 \\ \hline \end{array}$$

The University of Georgia

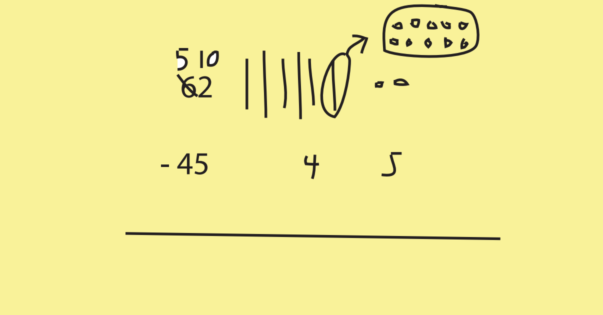
# Understanding and explaining the standard subtraction algorithm

## Grade 2



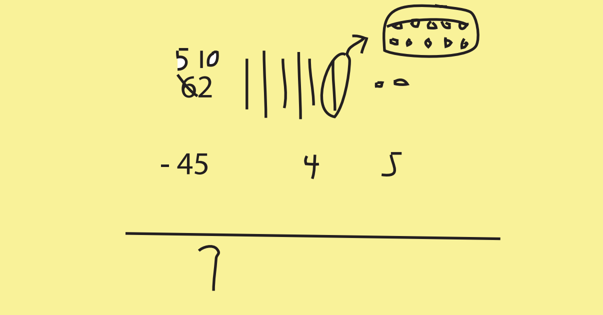
# Understanding and explaining the standard subtraction algorithm

## Grade 2



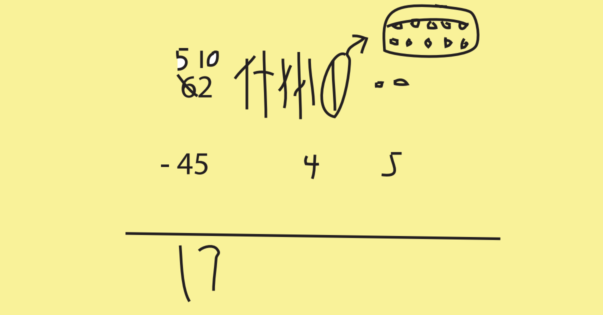
# Understanding and explaining the standard subtraction algorithm

## Grade 2

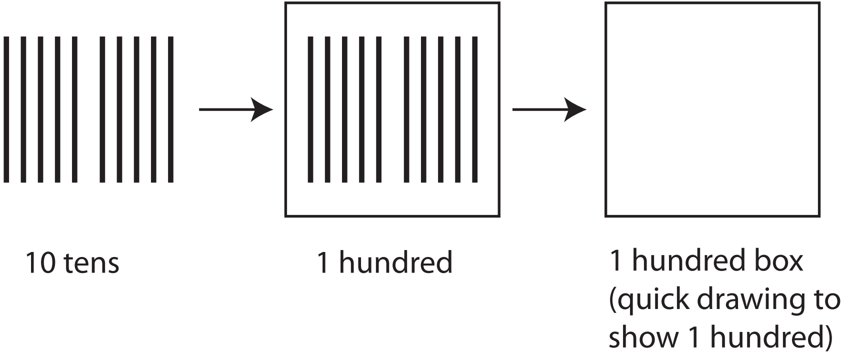


# Understanding and explaining the standard subtraction algorithm

## Grade 2



# Grade 2: Mathematical drawing to represent 1 hundred as 10 tens



# Your turn: Discuss multi digit addition and subtraction

Please see your handout!

Discuss the multi digit addition and subtraction methods

- Notice the advantages that written methods D and E for addition offer
- Notice the advantages of written method A over method B for subtraction

# Thank you!

Questions? Comments?

Find me on Twitter at SybillaBeckmann

Join the Mathematics Teaching Community online at https://mathematicsteachingcommunity.math.uga.edu/

Some resources:

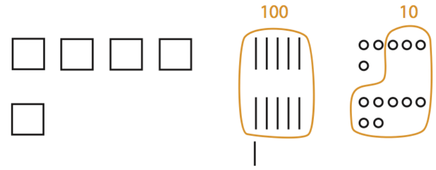
- Fuson, K.C. & Beckmann, S. (Fall/Winter 2012-02013). Standard algorithms in the Common Core State Standards. *National Council of of Supervisors of Mathematics Journal of Mathematics Education Leadership, 14*(2), 4 – 30. www.mathedleadership.org
- The Progressions for the Common Core State Standards: http://ime.math.arizona.edu/progressions/
- *Mathematics for Elementary Teachers* by Sybilla Beckmann

Adapted from: Fuson, K. C. & Beckmann, S. (Fall/Winter 2012-2013). Standard algorithms in the Common Core State Standards. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership,14(2)*, 4 - 30. www.mathedleadership.org

## Multidigit Addition Methods

### Place value drawing for all methods

456
+ 167




### General methods for 2- and 3-digit numbers

**D.**
456
+ 167
500
110
13
623

**E.**
456
+ 167
1 1
623

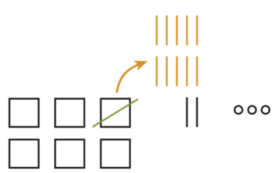
**F.**
5 6
~~4~~~~5~~6
+ 167
623

**G.**
1 1
456
+ 167
623

## Multidigit Subtraction Methods

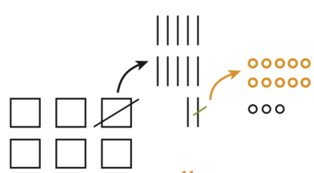
### Method A. Ungroup where needed first, then subtract

**1.** Ungroup hundreds



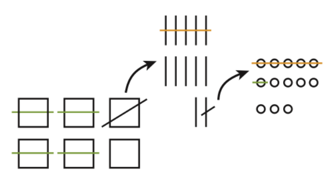
5 12
~~6~~~~2~~3
- 456

**2.** Ungroup tens



11
5 ~~12~~ 13
~~6~~~~2~~~~3~~
- 456

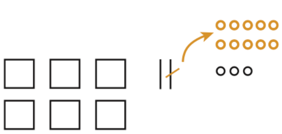
**3.** Subtract everywhere (in either direction)



5 11 13
~~6~~~~2~~~~3~~
- 456
167

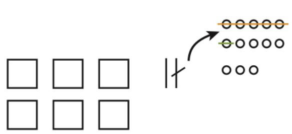
### Method B. Alternate ungrouping and subtracting for each column

**1.** Ungroup tens



1 13
6~~2~~~~3~~
- 456

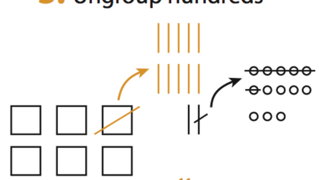
**2.** Subtract ones



1 13
6~~2~~~~3~~
- 456
47

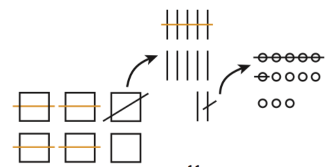
(Error)

**3.** Ungroup hundreds



11
5 ~~1~~ 13
~~6~~~~2~~~~3~~
- 456
7

**4.** Subtract tens, then hundreds



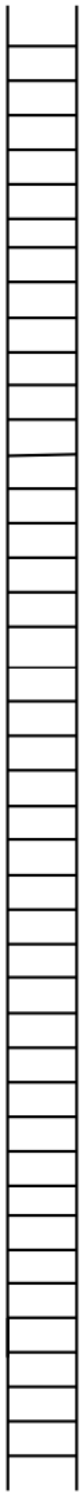
11
5 ~~1~~ 13
~~6~~~~2~~~~3~~
- 456
167

From: Fuson, K. C. & Beckmann, S. (2013). Teaching Phases and Standard Algorithms in the Common Core. Presentation to the National Council of Supervisors of Mathematics Annual Meeting. Find the full paper at www.math.uga.edu/~sybilla

# Learning Path for Multidigit Computation in CCSS

Bridging for teachers and students by coherent learning supports

Learning Path



**Phase 3: Compact methods for fluency**

Students use any good variation of writing the standard algorithm with no drawing to build fluency. They explain occasionally to retain meanings.

Math Sense-Making ↕ Math Drawings
Math Structure ↕ Math Explaining

**Phase 2: Research-based mathematically-desirable and accessible methods in the middle for understanding and growing fluency**

Students focus on and compare efficient, accurate, and generalizable methods, relating these to visual models and explaining the methods. They write methods in various ways and discuss the variations. They may use a helping step method for understanding and/or accuracy. They choose a method for fluency and begin solving with no drawing.

Math Sense-Making ↕ Math Drawings
Math Structure ↕ Math Explaining

**Phase 1: Students' methods elicited for understanding but move rapidly to Phase 2**

Students develop, discuss, and use efficient, accurate, and generalizable methods and other methods. They use concrete models or drawings that they relate to their written method and explain the reasoning used.

Note. Students may consider problems with special structure (e.g., 98 + 76) and devise quick methods for solving such problems. But the major focus must be on general problems and on generalizable methods that focus on single-digit computations (i.e., that are or will generalize to become a variation of writing the standard algorithm).